Acta Sci. Math., 38 (1976), 83—96

Factorization of operator pencils

HEINZ LANGER

Let L be the pencil

$$(1)\qquad L(\lambda) = \lambda^n I + \lambda^{n-1} L_{n-1} + \ldots + \lambda L_1 + L_0$$

where the coefficients $L_0, L_1, \ldots, L_{n-1}$ are bounded operators in a Banach space $\mathfrak{B}$. We set

$$(2)\qquad \mathbf{L} = \begin{bmatrix} -L_{n-1} & -L_{n-2} & \ldots & -L_1 & -L_0 \\ I & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{bmatrix};$$

this is an operator in $\mathbf{B} = \mathfrak{B}^n$. In the papers [1]—[5] it was shown that there is a close connection between certain invariant subspaces of $\mathbf{L}$ and the representation of $L(\lambda)$ as the product of a pencil of degree $n-1$ and of another of degree 1 with leading coefficients I. The aim of this note is the study of a similar connection for other types of factorizations, e.g. for those into factors of degree >1 each.

If the underlying space is a Hilbert space, $\mathfrak{B} = \mathfrak{H}$, and the coefficients in (1) are selfadjoint, then the invariant subspaces of $\mathbf{L}$ we shall treat are maximal nonnegative or maximal nonpositive with respect to some indefinite scalar product on $\mathbf{H} = \mathfrak{H}^n$.

Some of the results may turn out to be new even in the case of a matrix pencil. Theorem 4, for instance, states that in a unitary space a pencil of degree n with hermitean matrix coefficients can be written as the product of two pencils of degrees $\left[\frac{n}{2}\right]$ and $\left[\frac{n+1}{2}\right]$, such that one factor is invertible in the open upper and the other is invertible in the open lower half plane. [1])

Received October 1, 1974, revised January 14, 1975.

[1]) The author expresses his sincere thanks to A. S. Markus and J. Bognár for useful suggestions. To J. Bognár he is also indebted for a rewriting of the original manuscript in real English.

1. Preliminaries

In the following all the operators are bounded. If $\mathfrak{H}$ is a Hilbert space with scalar product $(.,.)$ and G is a selfadjoint operator in $\mathfrak{H}$, we define the G-scalar product $[.,.]$ by the equation

$$[x, y] = (Gx, y) \quad (x, y \in \mathfrak{H}). \tag{3}$$

We only need the case where G is indefinite and boundedly invertible, that is, in the terminology of [6] $\mathfrak{H}$ is a Krein space with respect to the scalar product (3). A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called G-nonnegative, G-nonpositive or G-neutral according as $[x, x] \geqq 0$, $\leqq 0$ or $=0$ for all $x \in \mathfrak{L}$; it is called uniformly G-positive if $[x, x] \geqq \gamma \|x\|^2$ for all $x \in \mathfrak{L}$ with some $\gamma > 0$. A G-nonnegative subspace which is not properly contained in any other G-nonnegative subspace is called maximal G-nonnegative. An operator A in $\mathfrak{H}$ is said to be G-selfadjoint, if $GA = (GA)^*$, or equivalently, if

$$[Ax, y] = [x, Ay] \quad \text{for all} \quad x, y \in \mathfrak{H}.$$

If $\mathfrak{L} \subset \mathfrak{H}$, we write

$$\mathfrak{L}^{[\perp]} = \{x : [x, \mathfrak{L}] = \{0\}\} \tag{4}$$

and call $\mathfrak{L}^{[\perp]}$ the G-orthogonal companion of $\mathfrak{L}$.

With the pencil (1) in the Banach space $\mathfrak{B}$ we associate the operators $\mathbf{L}$ (see (2)) and

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & \dots & 0 & I \\ 0 & 0 & \dots & I & L_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & I & \dots & L_3 & L_2 \\ I & L_{n-1} & \dots & L_2 & L_1 \end{bmatrix} \tag{5}$$

in $\mathbf{B} = \mathfrak{B}^n$. Evidently, $\mathbf{G}$ is boundedly invertible. If $\mathfrak{B} = \mathfrak{H}$ is a Hilbert space and the L_j are selfadjoint, then $\mathbf{G}$ is also selfadjoint; moreover, $\mathbf{G}$ is indefinite and $\mathbf{L}$ is $\mathbf{G}$-selfadjoint.

In the next section together with the Banach space $\mathfrak{B}$ we consider its dual $\mathfrak{B}^*$. Then (f^*, x) denotes the value of $f^* \in \mathfrak{B}^*$ at the point $x \in \mathfrak{B}$, and for a subspace $\mathfrak{L} \subset \mathfrak{B}$ we write

$$\mathfrak{L}^{\perp} = \{f^* \in \mathfrak{B}^* : (f^*, \mathfrak{L}) = \{0\}\}.$$

2. Invariant subspaces of L and factorizations of L.

Theorem 1. *The pencil* (1) *admits a factorization*

$$L(\lambda) = \tilde{M}(\lambda) K(\lambda) \tag{6}$$

with a pencil K: $K(\lambda) = \lambda^k I + \lambda^{k-1} K_{k-1} + \dots + \lambda K_1 + K_0$ *of degree* $k (<n)$ *and a pencil* $\tilde{M}$

of degree $n-k$ if and only if the operator $\mathbf{L}$ *in* $\mathfrak{B}$ *has an invariant subspace* $\mathfrak{K}$ *of the form*

$$\mathfrak{K}=\left\{\begin{bmatrix} K_{11}x_1+K_{12}x_2+\ldots+K_{1k}x_k \\ K_{21}x_1+K_{22}x_2+\ldots+K_{2k}x_k \\ \vdots \\ K_{n-k,1}x_1+K_{n-k,2}x_2+\ldots+K_{n-k,k}x_k \\ x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} : x_1, x_2, \ldots, x_k \in \mathfrak{B}\right\} \tag{7}$$

with bounded linear operators K_{ij} *in* $\mathfrak{B}$ *such that* $-K_{n-k,j}=K_{k-j}$ $(j=1, 2, \ldots, k)$. *In this case also the operators* K_{ij} $(1\leqq i\leqq n-k-1;\ 1\leqq j\leqq k)$ *are uniquely determined by the operators* $K_0, K_1, \ldots, K_{k-1}$. *Moreover,* $\sigma(K)=\sigma(\mathbf{L}|\mathfrak{K})$.[2])

Proof. (α) In order to show that the invariance of the subspace (7) implies the existence of a factorization (6), we consider the resolvent $(\mathbf{L}-\lambda\mathbf{I})^{-1}$ for sufficiently large λ. It has the following matrix form:

$$-\begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} L(\lambda)^{-1}[1, \lambda, \ldots, \lambda^{n-2}, \lambda^{n-1}] \begin{bmatrix} I & L_{n-1} & \ldots & L_2 & L_1 \\ 0 & I & \ldots & L_3 & L_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I & L_{n-1} \\ 0 & 0 & \ldots & 0 & I \end{bmatrix} +$$

$$+\begin{bmatrix} 0 & I & \lambda I & \ldots & \lambda^{n-3}I & \lambda^{n-2}I \\ 0 & 0 & I & \ldots & \lambda^{n-4}I & \lambda^{n-3}I \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I & \lambda I \\ 0 & 0 & 0 & \ldots & 0 & I \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

which can be verified by multiplication with the matrix of $\mathbf{L}-\lambda\mathbf{I}$ from the left. Applying $(\mathbf{L}-\lambda\mathbf{I})^{-1}$ to an element of (7) with $x_2=x_3=\ldots=x_k=0$, the $(n-k)$-th component is

$$\lambda^k L(\lambda)^{-1}\tilde{M}(\lambda)x_1+x_1, \tag{8}$$

[2]) For the definition of the spectrum of a pencil see e.g. [3].

where

$$\tag{9} \tilde{M}(\lambda) = [1, \lambda, \dots, \lambda^{n-2}, \lambda^{n-1}] \begin{bmatrix} I & L_{n-1} & \dots & L_2 & L_1 \\ 0 & I & \dots & L_3 & L_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & L_{n-1} \\ 0 & 0 & \dots & 0 & I \end{bmatrix} \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n-k,1} \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By the invariance of $\mathfrak{R}$, the element (8) equals $K_{n-k,1}y_1 + \dots + K_{n-k,k}y_k$ with

$$y_j = -\lambda^{k-j} L(\lambda)^{-1} \tilde{M}(\lambda) x_1, \quad j = 1, 2, \dots, k.$$

This gives $(\lambda^k I - \lambda^{k-1} K_{n-k,1} - \dots - K_{n-k,k}) L(\lambda)^{-1} \tilde{M}(\lambda) = I$, and the factorization (6) follows.

(β) Suppose, conversely, that $L(\lambda)$ has a factorization of the form (6). Define matrices $\mathscr{K}_j$ ($k \leqq j \leqq n-1$) by the equations

$$\mathscr{K}_j = \underbrace{\begin{bmatrix} -K_{k-1} & -K_{k-2} & \dots & -K_1 & -K_0 & 0 \dots 0 \\ I & 0 & \dots & 0 & 0 & 0 \dots 0 \\ 0 & I & \dots & 0 & 0 & 0 \dots 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 \dots I \end{bmatrix}}_{j \text{ columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \\ a \\ a \end{bmatrix}}\right\} j+1 \text{ rows}$$

Here $K_0, K_1, \dots, K_{k-1}$ are the coefficients of the factor $K(\lambda)$ appearing in (6). The first step in the partial division of a polynomial $\lambda^l B_l + \lambda^{l-1} B_{l-1} + \dots + \lambda B_1 + B_0$ by $K(\lambda)$ ($l \geqq k$) from the right gives a remainder whose coefficients are the entries of the product

$$[B_l, B_{l-1}, \dots, B_1, B_0] \mathscr{K}_l.$$

Therefore, the factorization (6) yields

$$[L_{n-1} - K_{k-1}, L_{n-2} - K_{k-2}, \dots, L_{n-k} - K_0, L_{n-k-1}, \dots, L_0] \mathscr{K}_{n-1} \dots \mathscr{K}_k = 0.$$

This is equivalent to

$$\tag{10} \mathbf{L} \mathscr{K}_{n-1} \dots \mathscr{K}_k = \begin{bmatrix} -K_{k-1} & \dots & -K_0 & 0 \dots 0 & 0 \\ I & \dots & 0 & 0 \dots 0 & 0 \\ \vdots & & \vdots & \vdots \quad \vdots & \vdots \\ 0 & \dots & 0 & 0 \dots I & 0 \end{bmatrix} \mathscr{K}_{n-1} \dots \mathscr{K}_k =$$

$$= \mathscr{K}_{n-1} \dots \mathscr{K}_k \begin{bmatrix} -K_{k-1} & \dots & -K_1 & -K_0 \\ I & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix},$$

which implies the invariance of the subspace $\mathscr{K}_{n-1}\dots\mathscr{K}_k\mathfrak{B}^k$ under $\mathbf{L}$. On the other hand, it is easy to see that this subspace has the form (7) with $K_{n-k,j}=-K_{k-j}$ $(j=1,2,\dots,k)$.

(γ) Suppose now again that the subspace $\mathfrak{K}$ of the form (7) is invariant under $\mathbf{L}$, and let $\mathbf{y}\in\mathfrak{K}$. Considering the components with index $n-k, n-k-1, \dots, 2$ of $\mathbf{Ly}$, one finds easily that the operators K_{ij} $(1\leqq i\leqq n-k-1;\ 1\leqq j\leqq k)$ are uniquely determined by $K_{n-k,1},\dots,K_{n-k,k}$. Therefore,

$$\mathfrak{K}=\mathscr{K}_{n-1}\dots\mathscr{K}_k\mathfrak{B}^k. \tag{11}$$

From (10), using the notation

$$\mathbf{K}=\begin{bmatrix}-K_{k-1}\dots-K_1 & -K_0\\ I\quad\dots\quad 0 & 0\\ \vdots\qquad\quad\vdots & \vdots\\ 0\quad\dots\quad I & 0\end{bmatrix},$$

we get

$$(\mathbf{L}-\lambda\mathbf{I})\mathscr{K}_{n-1}\dots\mathscr{K}_k=\mathscr{K}_{n-1}\dots\mathscr{K}_k(\mathbf{K}-\lambda\mathbf{I}),$$

and the last statement of the theorem follows. In the case $k=1$ Theorem 1 coincides with Lemma 2 of [5].

Taking adjoints [3]) in (6), we find

$$L^*(\lambda)=K^*(\lambda)\tilde{M}^*(\lambda),$$

where e.g. L^* denotes the pencil

$$L^*(\lambda)=\lambda^nI+\lambda^{n-1}L_{n-1}^*+\dots+\lambda L_1^*+L_0^*$$

in $\mathfrak{B}^*$. Therefore, by Theorem 1, the factor $\tilde{M}^*(\lambda)$ of $L^*(\lambda)$ corresponds to a subspace $\tilde{\mathfrak{M}}^*\subset\mathfrak{B}^*$, of the form

$$\tilde{\mathfrak{M}}^*=\left\{\begin{bmatrix}\tilde{M}_{11}^*f_1^*+\dots+\tilde{M}_{1,n-k}^*f_{n-k}^*\\ \vdots\\ \tilde{M}_{k1}^*f_1^*+\dots+\tilde{M}_{k,n-k}^*f_{n-k}^*\\ f_1^*\\ \vdots\\ f_{n-k}^*\end{bmatrix}:f_1^*,\dots,f_{n-k}^*\in\mathfrak{B}^*\right\}, \tag{12}$$

which is invariant under the operator

$$\tilde{\mathbf{L}}=\begin{bmatrix}-L_{n-1}^* & -L_{n-2}^*\dots-L_1^* & -L_0^*\\ I & 0\quad\dots\quad 0 & 0\\ 0 & I\quad\dots\quad 0 & 0\\ \vdots & \vdots & \\ 0 & 0\quad\dots\quad I & 0\end{bmatrix}$$

[3]) If $\mathfrak{B}$ is a Hilbert space, we also have to make a substitution $\lambda\to\lambda^*$ (λ^* — complex conjugate of λ).

in $\mathfrak{B}^* = (\mathfrak{B}^*)^n$. We are going to prove that the pair of subspaces $\mathfrak{K} \subset \mathfrak{B}$, $\tilde{\mathfrak{M}}^* \subset \subset \mathfrak{B}^*$, associated with the factorization (6), satisfies the relation

$$\tilde{\mathfrak{M}}^* = (\mathbf{G}\mathfrak{K})^\perp \quad \text{(see (5))}. \tag{13}$$

To this end first observe that, as one easily checks, $(\mathbf{GL})^* = \mathbf{G}^*\tilde{\mathbf{L}}$. Therefore, $(\mathbf{G}\mathfrak{K})^\perp$ is invariant under $\tilde{\mathbf{L}}$:

$$(\tilde{\mathbf{L}}(\mathbf{G}\mathfrak{K})^\perp, \mathbf{G}\mathfrak{K}) = ((\mathbf{GL})^*(\mathbf{G}\mathfrak{K})^\perp, \mathfrak{K}) = ((\mathbf{G}\mathfrak{K})^\perp, \mathbf{GL}\mathfrak{K}) \subset ((\mathbf{G}\mathfrak{K})^\perp, \mathbf{G}\mathfrak{K}) = \{0\}.$$

Further, an element

$$\begin{bmatrix} v_1^* \\ \vdots \\ v_k^* \\ f_1^* \\ \vdots \\ f_{n-k}^* \end{bmatrix} \in \mathfrak{B}^*; \quad v_1^*, \ldots, v_k^*, f_1^*, \ldots, f_{n-k}^* \in \mathfrak{B}^*,$$

belongs to $(\mathbf{G}\mathfrak{K})^\perp$ if and only if

$$\begin{bmatrix} K_{11}^* \ldots K_{n-k,1}^* & I & 0 \ldots 0 \\ K_{12}^* \ldots K_{n-k,2}^* & 0 & I \ldots 0 \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad \vdots \\ K_{1k}^* \ldots K_{n-k,k}^* & 0 & 0 \ldots I \end{bmatrix} \mathbf{G}^* \begin{bmatrix} v_1^* \\ \vdots \\ v_k^* \\ f_1^* \\ \vdots \\ f_{n-k}^* \end{bmatrix} = 0. \tag{14}$$

From (14), in view of (5), the vectors $v_1^*, \ldots, v_k^*$ can be expressed through $f_1^*, \ldots, f_{n-k}^*$, i.e., $(\mathbf{G}\mathfrak{K})^\perp$ has the form given on the right hand side of (12) with bounded operators $\tilde{M}_{ij}$ $(1 \leqq i \leqq k;\ 1 \leqq j \leqq n-k)$. It remains to show that $\tilde{M}_{k1}^*, \ldots, \tilde{M}_{k,n-k}^*$ are the coefficients of the pencil $\tilde{M}^*(\lambda)$. The first row in (14) expresses v_k^* through $f_1^*, \ldots, f_{n-k}^*$, thus it has the form

$$v_k^* - \tilde{M}_{k1}^* f_1^* - \ldots - \tilde{M}_{k,n-k}^* f_{n-k}^* = 0.$$

In order to get the factor of $L^*(\lambda)$, belonging to the subspace $(\mathbf{G}\mathfrak{K})^\perp$, we have to make the formal substitutions $v_k^* \to \lambda^{n-k}, f_1^* \to \lambda^{n-k-1}, \ldots, f_{n-k}^* \to 1$, so this factor is given by

$$[K_{11}^*, \ldots, K_{n-k,1}^*, I, 0, \ldots, 0]\,\mathbf{G}^* \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda^{n-k} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}. \tag{15}$$

But it is easy to see that the first $n-k+1$ components of

$$\mathbf{G}^* \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda^{n-k} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & \dots & 0 & 0 \\ L_{n-1}^* & I & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ L_2^* & L_3^* & \dots & I & 0 \\ L_1^* & L_2^* & \dots & L_{n-1}^* & I \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}$$

are the same, that is, the pencil (15) coincides with $\tilde{M}^*(\lambda)$ of (9), and (13) is proved.

In the rest of the paper $\mathfrak{B}$ will be a Hilbert space, $\mathfrak{B}=\mathfrak{H}$, and the operators L_j will be selfadjoint, $L_j=L_j^*$ $(0\leqq j\leqq n-1)$. Then the operator $\mathbf{L}$ is $\mathbf{G}$-selfadjoint [cf. (2) and (5)]. Further, the orthogonal complement appearing in (13) is to be taken with respect to the natural scalar product of $\boldsymbol{\mathfrak{H}}=\mathfrak{H}^n$, and (13) can also be written as as $\tilde{\mathfrak{M}}^*=\mathfrak{K}^{[\perp]}$ (see (4)), i.e. the subspaces $\mathfrak{K}$ and $\tilde{\mathfrak{M}}^*$ associated with the factorization (6) are $\mathbf{G}$-orthogonal companions of each other.

Theorem 2. *The pencil* (1) *with selfadjoint coefficients $L_0, L_1, \dots, L_{n-1}$ in the Hilbert space $\mathfrak{H}$ admits a factorization*

$$L(\lambda) = M^*(\lambda)R(\lambda)K(\lambda)$$

into three pencils with leading coefficient I and of degree m, r, k $(m+r+k=n)$ if and only if the operator $\mathbf{L}$ in $\boldsymbol{\mathfrak{H}}$ has a pair of invariant subspaces $\mathfrak{K}$ and $\mathfrak{M}\subset\mathfrak{K}^{[\perp]}$ of the form (7) *and*

$$\mathfrak{M} = \left\{ \begin{bmatrix} M_{11}x_1+\dots+M_{1m}x_m \\ \vdots \\ M_{n-m,1}\,x_1+\dots+M_{n-m,m}\,x_m \\ x_1 \\ \vdots \\ x_m \end{bmatrix} : x_1, x_2, \dots, x_m \in \mathfrak{H} \right\}$$

with bounded linear operators M_{ij} in $\mathfrak{H}$. The subspace $\mathfrak{M}$ and the pencil M are connected in the same way as $\mathfrak{K}$ and K.

Proof. Suppose $\mathbf{L}$ has a pair of invariant subspaces $\mathfrak{K}$, $\mathfrak{M}$ with the properties mentioned in the theorem. The condition $\mathfrak{M}\subset\mathfrak{K}^{[\perp]}$ implies

$$\text{(16)}\qquad [K_{11}^*, K_{21}^*, \dots, K_{n-k,1}^*, I, 0, \dots, 0]\mathbf{G} \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1m} \\ \vdots & & & \\ M_{n-m,1} & M_{n-m,2} & \dots & M_{n-m,m} \\ I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} = 0.$$

If the matrices $\mathscr{M}_j$ $(m \leqq j \leqq n-1)$ correspond to the subspace $\mathfrak{M}$ in the same way as the matrices $\mathscr{K}_j$ correspond to $\mathfrak{K}$ (cf. (11)), we get from (16)

$$[K_{11}^*, K_{21}^*, \ldots, K_{n-k,1}^*, I, 0, \ldots, 0]\mathbf{G}\mathscr{M}_{n-1} \ldots \mathscr{M}_m = 0. \tag{17}$$

But the components of the vector

$$[K_{11}^*, K_{21}^*, \ldots, K_{n-k,1}^*, I, 0, \ldots, 0]\mathbf{G}$$

are the coefficients of $\tilde{M}^*(\lambda)$ (see (9)). Therefore (17) yields that the subspace $\mathfrak{M} = \mathscr{M}_{n-1} \ldots \mathscr{M}_m \mathfrak{B}^m$ is invariant under the operator $\tilde{\mathbf{M}}$ in $\mathfrak{B}^{n-k}$ corresponding to the pencil $\tilde{M}^*(\lambda)$ (cf. (10)). Hence, by theorem 1, the polynomial $\tilde{M}^*(\lambda)$ has the right hand factor $M(\lambda)$. This reasoning can be reversed, and the statement follows from Theorem 1.

3. Maximal G-nonnegative invariant subspaces and factorizations

Let $\mathfrak{H}$ be a Hilbert space, and let $L_0, L_1, \ldots, L_{n-1}$ be selfadjoint operators in $\mathfrak{H}$. In order to show the existence of invariant subspaces of $\mathbf{L}$ of the form (7), we use results from operator theory in spaces with an indefinite metric. The key is

Theorem 3. *A maximal* **G**-*nonnegative* (**G**-*nonpositive*) *subspace* $\mathfrak{K} \subset \mathfrak{H}$ *which is invariant under the operator* $\mathbf{L}$ *has the form* (7) *with* $k = \left[\frac{n+1}{2}\right]$ $\left(k = \left[\frac{n}{2}\right]\right)$.

Proof. We restrict ourselves to the case $n = 2k$. Let $\mathfrak{K}$ be maximal $\mathbf{G}$-nonnegative and invariant under $\mathbf{L}$. Suppose $\mathfrak{K}$ contains a sequence of elements

$$x^{(r)} = \begin{bmatrix} x_1^{(r)} \\ x_2^{(r)} \\ \vdots \\ x_n^{(r)} \end{bmatrix}; \quad r = 1, 2, \ldots,$$

with $x_j^{(r)} \to 0$ $(r \to \infty;\ j = k+1, \ldots, n)$ and $\sum_{j=1}^{k} \|x_j^{(r)}\|^2 = 1$ $(r = 1, 2, \ldots)$. Then we have $[\mathbf{x}^{(r)}, \mathbf{x}^{(r)}] \to 0$ and, by Schwarz's inequality, $[\mathbf{L}^{\varkappa}\mathbf{x}^{(r)}, \mathbf{x}^{(r)}] \to 0$ $(r \to \infty;\ \varkappa = 1, 2, \ldots)$. Now from the matrix representation of $\mathbf{GL}, \mathbf{GL}^3, \ldots, \mathbf{GL}^{n-1}$ it easily follows that $x_j^{(r)} \to 0$ $(j = k, k-1, \ldots, 1)$. Contradiction. Therefore, if $\mathbf{x} \in \mathfrak{K}$, the relations $x_j = 0$ $(k+1 \leqq j \leqq n)$ imply $x_j = 0$ $(1 \leqq j \leqq k)$. Now it is easy to see that for an $\mathbf{x} \in \mathfrak{K}$ the first k components are uniquely defined by the last k components. Moreover, from the linearity of $\mathfrak{K}$ and the foregoing consideration, they are even bounded linear functions of the last k components. Therefore, with $\mathbf{x}' = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_{2k} \end{bmatrix} \in \mathfrak{H}^k$ and a bounded linear operator $\mathbf{K}'$ in $\mathfrak{H}^k$, the subspace $\mathfrak{K}$ can be written as

$$\mathfrak{K} = \left\{ \begin{pmatrix} \mathbf{K}'\mathbf{x}' \\ \mathbf{x}' \end{pmatrix} : \mathbf{x}' \in \mathfrak{D}' \right\},$$

where $\mathfrak{D}'$ is evidently a closed subspace of $\mathfrak{H}^k$. It remains to show that $\mathfrak{D}' = \mathfrak{H}^k$. Suppose $\mathfrak{D}' \neq \mathfrak{H}^k$. Then there exists an $\mathbf{y}'_0 \in \mathfrak{H}^k$ with the property

$$(\mathbf{G}'_{12}\mathfrak{D}', \mathbf{y}'_0) = \{0\}, \quad \text{where} \quad \mathbf{G}'_{12} = \begin{bmatrix} 0 \dots 0 & I \\ 0 \dots I & L_{n-1} \\ \vdots \quad \vdots & \vdots \\ I \dots L_{n-k+1} & L_{n-k} \end{bmatrix}.$$

This is equivalent to

$$\left[\mathfrak{K}, \begin{pmatrix} \mathbf{y}'_0 \\ 0 \end{pmatrix}\right] = \{0\}, \quad \text{that is} \quad \begin{pmatrix} \mathbf{y}'_0 \\ 0 \end{pmatrix} \in \mathfrak{K}^{[\perp]}.$$

But $\begin{pmatrix} \mathbf{y}'_0 \\ 0 \end{pmatrix}$ is **G**-neutral and does not belong to $\mathfrak{K}$, so that $\mathfrak{K}$ cannot be maximal **G**-nonnegative. Contradiction.

Theorems 1 and 3 yield a factorization of a selfadjoint pencil (1) as soon as the existence of a maximal **G**-nonnegative subspace $\mathfrak{K}$ of the corresponding operator **L** is known. From a result of PONTRJAGIN [6; Theorems IX. 7.2 and VIII. 3.2] we immediately obtain the following

Theorem 4. *Let L be a pencil in the finite dimensional unitary space $\mathfrak{H}$. Suppose that the coefficients of L are symmetric matrices. Decompose the nonreal spectrum σ_0 of L into two disjoint parts σ_I and σ_{II} such that $\sigma_I \cap \sigma_I^* = \emptyset$ [4]), $\sigma_{II} = \sigma_I^*$. Then L admits a factorization*

$$L(\lambda) = \tilde{L}_{II}(\lambda) L_I(\lambda), \tag{18}$$

where L_I and $\tilde{L}_{II}$ are pencils of degree $\left[\frac{n}{2}\right]$ and $\left[\frac{n+1}{2}\right]$. In addition, we may require that the nonreal spectrum of L_I and $\tilde{L}_{II}$ is σ_I and σ_{II} respectively.

Now let L be a pencil of the form (1) with selfadjoint coefficients L_j $(0 \leqq j \leqq n-1)$ in the infinite dimensional Hilbert space $\mathfrak{H}$. Suppose L has only real zeros (in the terminology of [3]), that is, each polynomial $p_x(\lambda) = (L(\lambda)x, x)$ $(x \in \mathfrak{H};\ x \neq 0)$ has n real zeros $\lambda_1(x) \geqq \lambda_2(x) \geqq \dots \geqq \lambda_n(x)$. Then the spectral zones

$$\Lambda_j = \{\lambda_j(x) : x \in \mathfrak{H}, x \neq 0\} \quad (j = 1, 2, \dots, n)$$

[4]) σ_I^* denotes the set of complex conjugates to the points of σ_I.

are intervals of the real axis, and the intersection of two different zones consists of no more than one point. Define

(19) $$\Lambda_+ = \bigcup_{j=0}^{\left[\frac{n-1}{2}\right]} \overline{\Lambda_{2j+1}}, \quad \Lambda_- = \bigcup_{j=1}^{\left[\frac{n}{2}\right]} \overline{\Lambda_{2j}}.$$

By [3; Theorem 2] the operator $\mathbf{L}$ is definitizable in $\mathfrak{H}$, so that [7; Theorem 3.2] it has a maximal $\mathbf{G}$-nonnegative and a maximal $\mathbf{G}$-nonpositive invariant subspace, say $\mathfrak{R}_+$ and $\mathfrak{R}_-$, with $\sigma(\mathbf{L}|\mathfrak{R}_+)\subset\Lambda_+$ and $\sigma(\mathbf{L}|\mathfrak{R}_-)\subset\Lambda_-$. From theorems 1 and 3 we obtain

Theorem 5. *Suppose that the pencil L of degree n in the Hilbert space $\mathfrak{H}$ has only real zeros. Then L admits a factorization of the form*

(20) $$L(\lambda) = \tilde{L}_-(\lambda)L_+(\lambda),$$

where L_+ and $\tilde{L}_-$ are pencils of degree $\left[\frac{n+1}{2}\right]$ and $\left[\frac{n}{2}\right]$, respectively, and $\sigma(L_+)\subset\Lambda_+$, $\sigma(\tilde{L}_-)\subset\Lambda_-$.

Taking adjoints in (20), we get a factorization

$$L(\lambda) = \tilde{L}_+(\lambda)L_-(\lambda),$$

where $L_-(\lambda)=\tilde{L}_-^*(\lambda)$, $\tilde{L}_+(\lambda)=L_+^*(\lambda)$. Evidently, degrees and spectra of L_- and $\tilde{L}_+$ have the same properties as those of $\tilde{L}_-$ and L_+. In general, the factorizations (18) and (20) are not uniquely determined by the properties listed in Theorems 4 and 5. A similar remark holds for Theorem 7 below.

4. A theorem on maximal G-nonnegative invariant subspaces

The following theorem is a slight generalization of [6, Theorem VIII. 3.2] as applied to bounded operators.

In the sequel, $\mathcal{S}_\infty$ denotes the set of compact operators in the Hilbert space $\mathfrak{H}$. If $B_2-B_1\in\mathcal{S}_\infty$ for two (bounded linear) operators B_1, B_2 in $\mathfrak{H}$, we write $B_1\sim B_2$. Clearly, $B_1\sim B_2$ is equivalent to $B_1^*\sim B_2^*$.

Theorem 6. *Suppose the operators A_1 and A_2 in $\mathfrak{H}$ have the following properties:*

(i) *A_1 is G_1-selfadjoint, A_2 is G_2-selfadjoint;*

(ii) *A_1 has a maximal uniformly G_1-positive invariant subspace $\mathfrak{L}_+$;*

(iii) $A_1\sim A_2$, $G_1\sim G_2$.

Then the nonreal spectrum σ_0 of A_2 is discrete. If the sets σ_I, σ_{II} form a partition of σ_0 with the properties $\sigma_I\cap\sigma_I^*=\emptyset$, $\sigma_{II}=\sigma_I^*$, then there exists a maximal G_2-nonnegative subspace $\mathfrak{R}_I$ which is invariant under A_2 and satisfies the conditions $\sigma(A_2|\mathfrak{R}_I)\cap\sigma_0=\sigma_I$,

$\sigma_{ess}(A_2|\mathfrak{K}_I)=\sigma_{ess}(A_1|\mathfrak{L}_+)$. *A similar statement holds with "G_2-nonnegative" replaced by "G_2-nonpositive".*

Proof. We start with four simple remarks which can be checked easily.

(a) If $(.,.)$ and $(.,.)_0$ are two equivalent Hilbert scalar products on $\mathfrak{H}$ and we have

$$(G_1x, y) = (G_1'x, y)_0, \quad (G_2x, y) = (G_2'x, y)_0$$

for all $x, y \in \mathfrak{H}$, then $G_1 \sim G_2$ implies $G_1' \sim G_2'$.

(b) If G_1', G_2' are boundedly invertible selfadjoint operators, then $G_1' \sim G_2'$ implies $|G_1'| \sim |G_2'|$ and $\operatorname{sgn} G_1' \sim \operatorname{sgn} G_2'$.

(c) Let A be a G-selfadjoint operator in the Hilbert space $\mathfrak{H}_0$ with scalar product $(.,.)_0$. Define a second Hilbert scalar product $(.,.)_1$ on $\mathfrak{H}_0$ by the equation

$$(x, y)_1 = (|G|x, y)_0 \quad (x, y \in \mathfrak{H}_0).$$

Denote the adjoints of A with respect to $(.,.)_0$ and $(.,.)_1$ by $A^\otimes$ and A^*, respectively. Then the condition $A \sim A^*$ is equivalent to $|G|A \sim A^\otimes|G|$.

(d) If A is G-selfadjoint, $G=P_+-P_-$ with two orthogonal projectors P_+, P_- $(P_++P_-=I)$ and $P_+AP_-\in\mathscr{S}_\infty$, then for each maximal G-nonnegative invariant subspace $\mathfrak{L}$ we have $\sigma_{ess}(A|\mathfrak{L})=\sigma_{ess}(P_+A|P_+\mathfrak{H})$.

Having made these remarks, consider the decomposition $\mathfrak{H}=\mathfrak{L}_+\dot{+}\mathfrak{L}_-$, where $\mathfrak{L}_-$ denotes the G_1-orthogonal companion of $\mathfrak{L}_+$. Introduce the Hilbert scalar product

$$(x, y)_0 = (G_1x_+, y_+)-(G_1x_-, y_-);$$

$$x = x_++x_-, \ y = y_++y_-; \quad x_+, y_+\in\mathfrak{L}_+; \quad x_-, y_-\in\mathfrak{L}_-.$$

Conditions (i) and (ii) imply

(21) $$A_1 = A_1^\otimes,$$

where $^\otimes$ denotes the adjoint with respect to the scalar product $(.,.)_0$. Define operators G_1', G_2' by the equations

$$(G_1x, y) = (G_1'x, y)_0, \quad (G_2x, y) = (G_2'x, y)_0 \quad (x, y\in\mathfrak{H}).$$

Then G_1' is the difference of two complementary projectors which are orthogonal with respect to the scalar product $(.,.)_0$. From (a) and (b) we have $G_1'\sim G_2'$, $|G_2'|\sim I$, therefore in view of (iii) and (21),

$$|G_2'|A_2-A_2^\otimes|G_2'| \sim A_2-A_2^\otimes \sim A_1-A_1^\otimes = 0,$$

i.e.

(22) $$|G_2'|A_2 \sim A_2^\otimes|G_2'|.$$

Introducing the scalar product $(x, y)_1=(|G_2'|x, y)_0$ in $\mathfrak{H}$, we have $(G_2x, y)=(G_2'x, y)_0= =(\operatorname{sgn} G_2'x, y)_1$. From remark (c) and (22) it follows that A_2 satisfies the conditions of [6; Theorem VIII. 3.2] relative to the decomposition of $\mathfrak{H}$ into eigenspaces of sgn G'_2. Therefore A_2 has a maximal G_2-nonnegative invariant subspace $\mathfrak{K}_I$ with the property $\sigma(A_2|\mathfrak{K}_I)\cap\sigma_0=\sigma_I$.

Moreover, for the projectors $P_1=\frac{1}{2}(I+G'_1)$, $P_2=\frac{1}{2}(I+\operatorname{sgn} G'_2)$ from (b) we obtain $P_1\sim P_2$ and $P_1A_1P_1\sim P_2A_2P_2$. By well-known results of perturbation theory, this implies that the operators $P_1A_1P_1$ and $P_2A_2P_2$ have the same essential spectrum. Then, with possible exception of the point zero, the same is true for the restrictions $P_1A_1|_{P_1\mathfrak{H}}$ and $P_2A_2|_{P_2\mathfrak{H}}$. All these considerations are invariant with respect to a shift $A_1\to A_1+\alpha I$, $A_2\to A_2+\alpha I$, α real. Therefore we find $\sigma_{\text{ess}}(P_1A_1|P_1\mathfrak{H})=\sigma_{\text{ess}}(P_2A_2|P_2\mathfrak{H})$. Now the last assertion of the theorem follows from (*d*).

5. Further factorization theorems

As an immediate consequence of Theorems 1, 3 and 6 we have

Theorem 7. *Let* L *be a pencil of the form* (1) *in the Hilbert space* $\mathfrak{H}$. *Assume that the following conditions are fulfilled:*

1) $L_j=L'_j+L''_j$, *where the operators* L'_j, L''_j *are selfadjoint* $(j=0, 1, \dots, n-1)$.

2) $L''_j\in\mathcal{S}_\infty$ $(j=0, 1, \dots, n-1)$.

3) *The pencil* $L^{(1)}: L^{(1)}(\lambda) = \lambda^n I+\sum_{j=0}^{n-1}\lambda^j L'_j$ *has only real zeros.*

4) *For the spectral zones* $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ *of* $L^{(1)}$ *we have*

$$\bar{\Lambda}_i\cap\bar{\Lambda}_j=\emptyset \quad (i\neq j;\ i,j=1,2,\dots,n).$$ [5])

Then the nonreal spectrum σ_0 *of* L *is discrete. If the sets* σ_I, σ_{II} *form a partition of* σ_0 *with the properties* $\sigma_I\cap\sigma_I^*=\emptyset$, $\sigma_{II}=\sigma_I^*$, *there exists a factorization* $L(\lambda)=\tilde{L}_{II}(\lambda)L_I(\lambda)$ *with two pencils* L_I, $\tilde{L}_{II}$ *of degree* $\left[\frac{n}{2}\right]$ *and* $\left[\frac{n+1}{2}\right]$ *such that the nonreal spectrum of* L_I *and* $\tilde{L}_{II}$ *is* σ_I *and* σ_{II}, *respectively, while* [6]) $\sigma_{\text{ess}}(L_I)=\sigma_{\text{ess}}(L)\cap\Lambda_+$, $\sigma_{\text{ess}}(\tilde{L}_{II})=\sigma_{\text{ess}}(L)\cap\Lambda_-$ (*cf.* (19)).

---

[5]) Condition 4) can be replaced by the weaker condition 4'): *The corresponding operator* $\mathbf{L}^{(1)}$ *in* $\mathfrak{H}$ *has no singular critical points* (see [2]).

[6]) The essential spectrum of a pencil L is by definition the essential spectrum of the corresponding operator $\mathbf{L}$ (see (2)).

Proof. We have to show that there exists a maximal **G**-nonnegative subspace invariant under **L**. But the operators

$$\mathbf{A}_2 = \mathbf{L}, \quad \mathbf{A}_1 = \mathbf{L}^{(1)} = \begin{bmatrix} -L'_{n-1} & \dots & -L'_1 & -L'_0 \\ I & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix},$$

$$\mathbf{G}_2 = \mathbf{G} = \begin{bmatrix} 0 & 0 & \dots 0 & I \\ 0 & 0 & \dots I & L_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ I & L_{n-1} & \dots L_1 & L_1 \end{bmatrix}, \quad \mathbf{G}_1 = \begin{bmatrix} 0 & 0 & \dots 0 & I \\ 0 & 0 & \dots I & L'_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ I & L'_{n-1} & \dots L'_2 & L'_1 \end{bmatrix}$$

satisfy the conditions of theorem 6 as soon as $\mathfrak{L}_+$ is chosen to be the Riesz spectral subspace of $\mathbf{L}_1$ belonging to Λ_+.

Obviously, Theorem 7 contains Theorem 4. Another example for the application of Theorem 7 is the following. Take $n=2$, $L'_0=\alpha I$ (α a real number), and suppose that conditions 1), 2) of theorem 7 are fulfilled. Then conditions 3) and 4) are also fulfilled if and only if

$$(L'_1x, x)^2 - 4\alpha\|x\|^4 \geqq \gamma\|x\|^4 \quad (x\in\mathfrak{H})$$

with some $\gamma>0$.

Theorem 8. *Let L be a pencil of odd degree $n=2k+1$ with selfadjoint coefficients $L_0, L_1, \dots, L_{2k}$. Suppose there exists a closed subset Ω of the open upper half plane such that each polynomial $p_x(\lambda)=(L(\lambda)x, x)$ ($x\in\mathfrak{H}$, $x\neq 0$) has one zero on the real axis and the other $2k$ zeros are in $\Omega\cup\Omega^*$. Then L admits a factorization of the form*

$$L(\lambda) = K^*(\lambda)(\lambda I - Z_0)K(\lambda), \tag{23}$$

where K is a pencil of degree k with leading coefficient I, the operator Z_0 is selfadjoint, and $\sigma(K)=\sigma(L)\cap\Omega$, $\sigma(Z_0)=\sigma(L)\cap R^1$.

Proof. The Riesz spectral subspace of **L** belonging to $\Omega\cap\sigma(L)$ is maximal **G**-neutral and maximal **G**-nonpositive. Therefore (23) follows from Theorems 2 and 3.

The existence of a factorization (6) with $k=1$, $K(\lambda)=\lambda I-K_0$ implies that K_0 is a solution of the equation

$$L(K_0) \equiv K_0^n + L_{n-1}K_0^{n-1} + \dots + L_1K_0 + L_0 = \frac{1}{2\pi i}\int_{\mathscr{C}_0} L(\lambda)(\lambda I - K_0)^{-1}\,d\lambda = 0$$

($\mathscr{C}_0$ — contour surrounding $\sigma(K_0)$). Conversely, every solution K_0 of $L(K_0)=0$ evidently gives rise to a factorization (6) with $k=1$ and $K(\lambda)=\lambda I-K_0$.

In the case $n=3$, under the hypotheses of Theorem 8 the equation $L(Z)=0$ has three solutions Z_1, Z_2, Z_3. They are uniquely determined by the properties

$$\sigma(Z_1) = \sigma(L)\cap\Omega, \sigma(Z_2) = \sigma(L)\cap\Omega^*, \sigma(Z_3) = \sigma(L)\cap R^1.$$

The operator Z_3 is similar to a selfadjoint operator. Indeed, the existence and the properties of Z_1 and Z_2 follow from Theorem 8, the existence of Z_3 from [8] (see also [9; Theorem 6]). We have the factorizations

$$L(\lambda) = (\lambda I - Z_1^*)(\lambda I + Z_1^* + Z_1 + L_2)(\lambda I - Z_1) = (\lambda I - Z_3^*)(\lambda I + Z_3^* + Z_1 + L_2)(\lambda I - Z_1)$$

and their analogs with Z_2 instead of Z_1. If L is of even degree $n=2k$ and $\sigma(L)\cap R^1=\emptyset$, then from Theorems 2 and 3 we get the well-known factorization

$$L(\lambda) = K^*(\lambda)K(\lambda),$$

where K is a pencil of degree k and $\sigma(K)=\sigma_0$, the part of $\sigma(L)$ located in the upper half plane. Moreover, K is uniquely determined by these properties as the corresponding subspace $\mathfrak{K}$ is uniquely determined by the properties $\sigma(\mathbf{L}|\mathfrak{K})=\sigma_0$, $\sigma(\mathbf{L}|\mathfrak{K}^{(\perp)})=\sigma_0^*$. Under the weaker condition $L(\lambda)\geqq 0$ $(\lambda=\bar{\lambda})$ a similar factorization was proved in [10; Theorem 3.3].

Literature

[1] M. G. Krein, H. Langer, On some mathematical principles of the linear theory of damped vibrations of continua, *Proceedings International Symposium on Applications of the Theory of Functions in Continuum Mechanics, Tbilisi 1963*, vol. 2, 283—322 (Moscow, 1965). [Russian]

[2] H. Langer, Über stark gedämpfte Scharen im Hilbertraum, *J. Math. Mech.*, **17** (1968), 685—705.

[3] H. Langer, Über eine Klasse polynomialer Scharen selbstadjungierter Operatoren im Hilbertraum. I, *J. Functional Analysis*, **12** (1973), 13—29.

[4] H. Langer, Über eine Klasse polynomialer Scharen selbstadjungierter Operatoren im Hilbertraum, II, *J. Functional Analysis*, **16** (1974), 221—234.

[5] H. Langer, Über eine Klasse nichtlinearer Eigenwertprobleme, *Acta Sci. Math.*, **35** (1973), 79—93.

[6] J. Bognár, *Indefinite inner product spaces*, Springer-Verlag (Berlin—Heidelberg—New York, 1974).

[7] H. Langer, Invariante Teilräume definisierbarer J-selbstadjungierter Operatoren, *Ann. Acad. Sci. Fenn.*, Ser. A I, **475** (1971), 1—23.

[8] V. I. Macajev, A. I. Virozub, On spectral properties of a class of selfadjoint operator functions, *Funkcional. Analiz*, **8:1** (1974), 1—10. [Russian]

[9] H. Langer, Zur Spektraltheorie polynomialer Scharen selbstadjungierter Operatoren, *Math. Nachrichten*, **65** (1975), 301—319.

[10] M. Rosenblum, J. Rovnyak, The factorization problem for non-negative operator valued functions, *Bull. AMS*, **77** (1971), 287—317.

SEKTION MATHEMATIK
TU DRESDEN
DDR—8027 DRESDEN